INDIVIDUAL TEST / ORAL EXAM
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied and Computational Mathematics

1. Let $f(x)$ defined on $[0,\ 1]$ be a smooth function with sufficiently many derivatives. $x_i = ih$, where $h = \frac{1}{N}$ and $i = 0, 1, \cdots, N$ are uniformly distributed points in $[0,\ 1]$. What is the highest integer k such that the numerical integration formula
(1)
$$I_N = \frac{1}{N}\left(a_0(f(x_0) + f(x_N)) + a_1(f(x_1) + f(x_{N-1})) + \sum_{i=2}^{N-2} f(x_i)\right)$$
is k-th order accurate, namely

$$\left| I_N - \int_0^1 f(x)dx \right| \leq Ch^k \tag{2}$$

for a constant C independent of h? Please describe the procedure to obtain the two constants a_0 and a_1 for this k.

2. The classical Euclidean Algorithm to find the greatest common divisor $\gcd(m, n)$ of two positive integers $m < n$ requires only $O(\log n)$ arithmetic operation. However, it uses division with a reminder, which is a rather slow operation. Your task is to design and analyze a **division-free** algorithm.

More precisely, using that for non-zero integers k and l we have
$$\gcd(2k, 2l) = 2\gcd(k, l),$$
$$\gcd(2k+1, 2l) = \gcd(2k+1, l),$$
$$\gcd(2k+1, l) = \gcd(2k+1-l, l)$$

- design an efficient algorithm to compute $\gcd(m, n)$ that uses only subtraction and division by 2 (the latter is very fast as it is equivalent to a shift of the bit representation of the operand);
- give a motivated estimate on the complexity of your algorithm.